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Special Algebraic Structures for Integrability by compensation in PDEs

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A Remark

Let $f \in L^1(\mathbb{D})$ and v be a weak solution of

$$\begin{cases} -\Delta v = f & \text{in } \mathbb{D} \\ v = 0 & \text{in } \partial\mathbb{D} \end{cases} \tag{1}$$

In general a solution v of (1) is neither in $L^\infty(\mathbb{D})$ nor in $W^{1,2}(\mathbb{D})$.
We can a-priori say that

$$\nabla v \in \cap_{p<2} W^{1,p}(\mathbb{D}) \quad \text{and} \quad v \in \cap_{q<+\infty} L^q(\mathbb{D}).$$

Moreover

$$v \in BMO(\mathbb{D}) \quad \text{and} \quad \nabla v \in L^{2,\infty}(\mathbb{D}).$$

The weak L^2 space:

$$L^{2,\infty}(\mathbb{D}) = \{f \text{ measurable} : \quad \sup_{\lambda \geq 0} \lambda |\{x \in \mathbb{D} \ : |f(x)| \geq \lambda\}|^{1/2} < +\infty\}$$

- It is a Banach Space, it is the dual of $L^{2,1}(\mathbb{D})$ space, the space of measurable functions f satisfying

$$2 \int_0^{+\infty} |\{x \in \mathbb{D} \ : |f(x)| \geq \lambda\}|^{1/2} d\lambda < +\infty.$$

- Recall that $\|f\|^2_{L^2(\mathbb{D})} = 2 \int_0^{+\infty} \lambda |\{x \in \mathbb{D} \ : |f(x)| \geq \lambda\}| d\lambda < +\infty$ and

$$L^{2,1}(\mathbb{D}) \subsetneq L^2(\mathbb{D}) \subsetneq L^{2,\infty}(\mathbb{D})$$

The Riesz Potential $\frac{1}{|x|} \in L^{2,\infty}(\mathbb{D}) \setminus L^2(\mathbb{D})$ and $\frac{1}{|x| \log(|x|^{-1})} \chi_{B(0,1/2)} \in L^2(\mathbb{D}) \setminus L^{2,1}(\mathbb{D})$.

Special case:

$$f = \nabla b \cdot \nabla^{\perp} a = \partial_x a \partial_y b - \partial_y a \partial_x b, \quad \nabla a, \nabla b \in L^2(\mathbb{D})$$

Theorem: [Wente, '69]

Let $\varphi \in W_0^{1,1}(\mathbb{D})$ be the solution

$$\begin{cases} -\Delta \varphi = \partial_x a \partial_y b - \partial_y a \partial_x b = \nabla^{\perp} a \cdot \nabla b, & \text{in } \mathbb{D} \\ \varphi = 0 & \text{on } \partial \mathbb{D}\,. \end{cases}$$

Then $\varphi \in W^{1,2} \cap C^0(\mathbb{D})$ and

$$\|\varphi\|_{L^\infty} + \|\nabla \varphi\|_{L^2} \leq C \|\nabla a\|_{L^2} \|\nabla b\|_{L^2}$$

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Theorem: [Wente, '69], [Tartar, '83]

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Then $\varphi \in W^{1,2} \cap C^0(\mathbb{D})$ and

$$\|\varphi\|_{L^\infty} + \|\nabla\varphi\|_{L^2} + \|\nabla\varphi\|_{L^{2,1}} \leq C\|\nabla a\|_{L^2}\|\nabla b\|_{L^2}$$

Underlining Idea:

Let $\nabla a, \nabla b \in L^2\left(\mathbb{R}^2\right)$, then

$$\|(\partial_x a \partial_y b - \partial_y a \partial_x b) * \log|x|\|_{L^\infty \cap \dot{W}^{1,2}(\mathbb{R}^2)} \leq C_0 \|\nabla a\|_{L^2} \|\nabla b\|_{L^2}.$$

- NOTE: Due to the special nature of the bilinear right-hand side one "gains" some integrability passing from BMO$(\mathbb{D})$ to $L^\infty(\mathbb{D})$ and from $L^{2,\infty}(\mathbb{D})$ to $L^2(\mathbb{D})$.

▶ Remarks on Jacobian structure [Coifman, Lions, Meyer and Semmes ('93)]

If $\nabla a, \nabla b \in L^2(\mathbb{R}^2)$, then

$$\nabla b \cdot \nabla^{\perp} a = \partial_x a \partial_y b - \partial_y a \partial_x b \in \mathcal{H}^1(\mathbb{R}^2)(= \text{Hardy Space})$$

where

$$\mathcal{H}^1(\mathbb{R}^2) = \{f \in L^1(\mathbb{R}^2) : \quad \int_{\mathbb{R}^2} \sup_{t>0} |\phi_t * f(x)| \, dx < +\infty\}$$

$\phi_t(x) := t^{-2} \, \phi(t^{-1}x)$, ϕ is some function in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ satisfying $\int_{\mathbb{R}^2} \phi(x) \, dx = 1$.

▶ The Hardy space is a strict subspace of L^1 featuring a good behavior when being convoluted with Calderon-Zygmund kernels.

Counterexample to Wente's Inequality in the Context of Neumann Boundary Conditions

Theorem [Da Lio & Palmurella, 2017], [Hirsch, 2017]

There are $a, b \in (L^\infty \cap H^1)(\mathbb{D})$ with $\int_{\mathbb{D}} \nabla^\perp b \cdot \nabla a \, dy = 0$ such that every solution $\varphi \in W^{1,1}(\mathbb{D})$ of:

$$\begin{cases} -\Delta\varphi = \nabla^\perp b \cdot \nabla a & \text{in } \mathbb{D}, \\ \partial_\nu \varphi = 0 & \text{on } \partial\mathbb{D} \end{cases}$$

is <u>neither</u> in $H^1(\mathbb{D})$ <u>nor</u> in $L^\infty(\mathbb{D})$,

.

Fractional counterpart of the CLMS estimate for the Jacobians

One main issue:

$$\nabla(ab) = \nabla a\, b + a\, \nabla b \quad \text{but} \quad (-\Delta)^{\frac{1}{4}}(ab) \neq (-\Delta)^{\frac{1}{4}} a\, b + a\, (-\Delta)^{\frac{1}{4}} b$$

Let us call the bilinear operator $\mathcal{C}_{\frac{1}{2}}(a, b)$ to be the error of the missing product rule. That is

$$\mathcal{C}_{\frac{1}{2}}(a, b) = (-\Delta)^{\frac{1}{4}}(ab) - (-\Delta)^{\frac{1}{4}} a\, b - a(-\Delta)^{\frac{1}{4}} b.$$

We can also write this operator in integral form:

$$\mathcal{C}_{\frac{1}{2}}(a, b)(x) = c \int_{\mathbb{R}} \frac{(a(x) - a(y))\ (b(x) - b(y))}{|x - y|^{1+\frac{1}{2}}}\, dy.$$

Three-term commutator estimate

Theorem [DL & Rivière, 2011]

Let $a, b \in C_c^\infty(\mathbb{R})$ then

$$\|(-\Delta)^{1/4}\mathcal{C}_{\frac{1}{2}}(a,b)\|_{\mathcal{H}^1(\mathbb{R})} \lesssim \|(-\Delta)^{\frac{1}{4}}a\|_{L^2}\ \|(-\Delta)^{\frac{1}{4}}b\|_{L^2}$$

► NOTE: The above estimate represents the borderline case of the so-called Leibniz-Rules for Fractional Derivatives due to [Kenig, Ponce, Vega (1993)] obtained in the L^p spaces.

An Application: Regularity of Weak Harmonic Maps into a Closed Submanifold $\mathcal{N} \hookrightarrow \mathbb{R}^m$

Critical points $u \in W^{1,2}(\mathbb{D}, \mathcal{N})$ of

$$\mathcal{L}(u) = \frac{1}{2} \int_{\mathbb{D}} |\nabla u|^2 dx^2$$

[Hélein (1989)]: Case $\mathcal{N} = S^{m-1} \quad \Rightarrow \quad$ they satisfy

$$-\Delta u^i = \sum_{j=1}^{m} [u^i \nabla u^j - u^j \nabla u^i] \cdot \nabla u^j \text{ with } \operatorname{div}(u^i \nabla u^j - u^j \nabla u^i) = 0$$

By Poincaré Lemma there is $B \in W^{1,2}(\mathbb{D})$ such that $\nabla^{\perp} B_{ij} = u^i \nabla u^j - u^j \nabla u^j$, ($\nabla^{\perp} B_{ij} = (-\partial_y B_{ij}, \partial_x B_{ij})$) and

$$-\Delta u^i = \sum_{j=1}^{m} (\partial_x B_{ij} \partial_y u^j - \partial_y B_{ij} \partial_x u^j)$$

General smooth manifolds $\mathcal{N}$

- Moving Frame Method, [Hélein, 1991],

- Consequence of regularity for Schrödinger Systems with Antisymmetric Potentials in 2-D, [Rivière, 2006]

Theorem [Rivière, 2006]

Let $u \in W^{1,2}(\mathbb{D}, \mathbb{R}^m)$ and $\Omega \in L^2(\mathbb{D}, \mathbb{R}^2 \otimes so(m))$ solve

$$\boxed{-\Delta u = \Omega \cdot \nabla u}$$

Then $u \in W^{1,p}_{loc}(B^2, \mathbb{R}^m), \ \ \forall p < +\infty.$

Remark:

The antisymmetry (which is an algebraic condition) and NOT the divergence-free structure (which a differential condition) of the potential is actually the essential structure in the regularity and compactness properties for conformally invariant variational problems in 2-D $\Rightarrow$ *it is a conceptual change in Integrability by Compensation.*

Heuristics: Why can we hope to have better regularity?
Example in 1-D:

$$\ddot{y} = \Omega \, \dot{y}$$

If Ω is antisymmetric then its eigenvalues are imaginary and the solution is bounded.

Antisymmetry in nonlocal framework

Theorem [DL & Rivière, 2011, Mazowiecka & Schikorra, 2018]

Let $v \in L^2(\mathbb{R}, \mathbb{R}^m)$ and $\Omega \in L^2(\mathbb{R}, so(m))$ solve

$$\boxed{(-\Delta)^{1/4} v = \Omega \cdot v}$$

Then $v \in L^p_{loc}(\mathbb{R})$ for $p < +\infty$.

Very Weak Solutions to Elliptic Systems in Divergence Form: New Compensation not based on Antisymmetry

Theorem [Jin, Maz'ya, Van Schaftingen, 2009]

There is $A \in C^o \cap W^{1,2}(\mathbb{D}, \mathrm{Sym}(n))$ such that for some $\lambda > 0$

$$\langle A\xi, \xi \rangle \geq \lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^m$$ strong ellipticity

A symmetric and there is $u \in W^{1,1}(\mathbb{D}) \hookrightarrow L^2(\mathbb{D})$ such that

$$\mathrm{div}(A\nabla u) = 0 \;\text{ i.e. }\; \partial_{x_i}(A_{ij}\partial_{x_i} u) = 0$$

but

$$u \notin W^{1,p}_{loc}(\mathbb{D}), \quad \forall p > 1.$$

Question asked by Brezis.

Elliptic Systems with Critical Involution

Theorem [DL, Rivière, 2019]

Let $A \in W^{1,2}(\mathbb{D}, \text{Sym}(n))$, such that $A^2 = Id_n$ and let $u \in L^2(\mathbb{D}, \mathbb{R}^n)$ solve

$$\text{div}\,(A\,\nabla u) = 0 \quad \text{in } \mathcal{D}'(D)$$

Then $u \in \bigcap_{p<2} W^{1,p}_{loc}(\mathbb{D}, \mathbb{R}^n)$.

► Observe that the system is elliptic with principal symbol $|\xi|^2 A$ but it is not strongly elliptic in the Legendre-Hadamard sense since A is not definite positive.

An Important Step: Bourgain-Brezis Inequality

Theorem [Bourgain & Brezis 2002]

Let u be a function in $\mathbb{R}^2$ such that $\nabla u \in (H^{-1} + L^1)(\mathbb{R}^2)$ Then

$$\|u - c\|_{L^2(\mathbb{R}^2)} \leq c\|\nabla u\|_{(H^{-1}+L^1)(\mathbb{R}^2)} \tag{2}$$

where

$$\|\nabla u\|_{(H^{-1}+L^1)(\mathbb{R}^2)} = \inf_{\substack{\nabla u = f+g \\ f\in H^{-1},\, g\in L^1}} (\|f\|_{H^{-1}} + \|g\|_{L^1})$$

Remark

Our proof is based on the observation that the kernel of $\partial^2_{\bar{z}\bar{z}}$ which corresponds to $\mathcal{F}^{-1}[\frac{1}{(\bar{\xi})^2}] \in L^\infty$ and not only in BMO as the kernel of $\partial^2_{z\bar{z}}$.

- same phenomena appear in the context of 2 dimensional Euler equations with nonnegative vorticity, [Delort, 1991], [Evans & Müller, 1994], [Semmes, 1994]

Boundary characterization of Hardy spaces

[Hardy, 1915]: For every $f\colon \mathbb{D} \to \mathbb{C}$ holomorphic

$$r \mapsto \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \text{ is non decreasing for every } p > 0$$

Hardy spaces, [Riez,1923]:

$$\mathcal{H}^p(\mathbb{D}) = \{f\colon \mathbb{D} \to \mathbb{C} : \ f \text{ holomorphic, } \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < +\infty\}.$$

If $f \in \mathcal{H}^p(\mathbb{D})$ then there $g \in L^p(S^1)$ exists and

$$\lim_{r\to 1^-} \int_0^{2\pi} |f(re^{i\theta}) - g(e^{i\theta})|^p d\theta = 0 \text{ as well as } \lim_{r\to 1^-} f(re^{i\theta}) = g(e^{i\theta}), \ \ a.e\ \theta.$$

Then one defines

$$\|f\|_{\mathcal{H}^p(\mathbb{D})} := \|g\|_{L^p(S^1)}.$$

Boundary characterization of Bergmann space

$$\mathcal{A}^2(\mathbb{D}) := \{f : \mathbb{D} \to \mathbb{C} : \ f \text{ holomorphic and } \|f\|_{L^2(\mathbb{D})} < +\infty\}$$

- One has
$$\mathcal{H}^1(\mathbb{D}) \hookrightarrow \mathcal{A}^2(\mathbb{D}) \quad \text{with} \quad \|f\|_{L^2(\mathbb{D})} \leq C\|f\|_{L^1(S^1)}.$$
- If $\lim_{r\to 1^-} \|f(re^{i\theta})\|_{H^{-1/2}(S^1)} < +\infty$ then the following inequality holds as well:
$$\|f\|_{L^2(\mathbb{D})} \leq C\|f\|_{H^{-1/2}(S^1)} := \lim_{r\to 1^-} \|f(re^{i\theta})\|_{H^{-1/2}(S^1)}$$

Bergmann-Bourgain-Brezis type inequality

Theorem [DL, Rivière, Wettstein 2022]

$f \in \mathcal{A}^2(\mathbb{D})$ if and only if

$$\|f\|_{L^1+H^{-1/2}(S^1)} := \limsup_{r\to 1^-} \|f(re^{i\theta})\|_{L^1+H^{-1/2}(S^1)} < +\infty.$$

Moreover, it holds

$$\frac{1}{C}\|f\|_{L^2(\mathbb{D})} \leq \|f\|_{L^1+H^{-1/2}(S^1)} \leq C\|f\|_{L^2(\mathbb{D})}.$$

$\Rightarrow$ $\|f\|_{L^1+H^{-1/2}(S^1)}$, $\|f\|_{H^{-1/2}(S^1)}$ and $\|f\|_{L^2(\mathbb{D})}$ define equivalent norms on the Bergmann space $\mathcal{A}^2(D)$.

Fractional Bourgain-Brezis type inequality

Theorem [DL, Rivière, Wettstein 2022]

Let $u \in \mathcal{D}'(S^1)$ be such that

$$(-\Delta)^{\frac{1}{4}} u,\ \mathcal{R}(-\Delta)^{\frac{1}{4}} u \in \dot{H}^{-\frac{1}{2}}(S^1) + L^1(S^1).$$

Then

$$\left\| u - \fint_{S^1} u \right\|_{L^2(S^1)} \leq C \left(\|(-\Delta)^{1/4} u\|_{\dot{H}^{-1/2}(S^1)+L^1(S^1)} + \|\mathcal{R}(-\Delta)^{1/4} u\|_{\dot{H}^{-1/2}(S^1)+L^1(S^1)} \right)$$

Remark in the L^1 case

- [Stein & Weiss, 1960] Estimate in L^1.
- [Schikorra, A; Spector, D; Van Schaftingen 2017] Both summands necessary for 1-D domains for the L^1 estimate namely $(-\Delta)^{\frac{1}{4}} u, \mathcal{R}(-\Delta)^{\frac{1}{4}} u \in L^1$.